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A Numerical Approach for Solving Some Convex Maximization Problems

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Abstract. We are concerned with concave programming or the convex maximization problem. In this paper, we propose a method and algorithm for solving the problem which are based on the global optimality conditions first obtained by Strekalovsky (Soviet Mathematical Doklady, 8(1987)). The method continues approaches given in (Journal of global optimization, 8(1996); Journal of Nolinear and convex Analyses 4(1)(2003)). Under certain assumptions a convergence property of the proposed method has been established. Some computational results are reported. Also, it has been shown that the problem of finding the largest eigenvalue can be found by the proposed method.

Key words: algorithm, concave programming, global maximizer, global optimization, simple set

1. Introduction

We consider the problem of maximizing a convex function over a simple set $D \subset \mathbb{R}^n$:

$$f(x) \longrightarrow \max, \quad x \in D. \tag{1.1}$$

Problem (1.1) has many applications in economics and engineering. Convex maximization (equivalently and concave programming) problem as a special case of problem (1.1), where D is a polytope, is originated from Tuy [15]. The cutting plane method for this case was proposed in [15]. There are many iterative algorithms [8,9,11,12] for solving the convex maximization problem by resorting to the cutting plane or branch and bound techniques as well as other methods. Problem (1.1) is multiextremal and from the computational complexity viewpoint it is NP-hard. Classical local optimality conditions can't guarantee always finding a global optimal solution. We consider, in this paper, a method and algorithm for solving problem (1.1) based on the global optimality condition in [14]. Our approach uses properties of the maximum function and the simple set.

The paper is organized as follows. In Section 2, we give existing optimality conditions and related method and algorithm for problem (1.1). Section 3 describes a method and algorithm based on the maximum function. In Section 4, we report computational experiments on the proposed algorithm.

2. A Maximization of Strongly Convex Function over a Simple Set

A notion of a simple set was introduced in [3] as follows.

DEFINITION 1. A set D in $\mathbb{R}^n$ is called a simple set if the following conditions hold:

(a) D is compact,
(b) the problem of maximizing a linear function over D is solvable by a "simple" method.

We say that condition (b) holds, for example, if it can be solved as a linear programming problem (i.e., if D is a convex polyhedron) or if an analytical form of the solution is explicitly given. For example, if D is a box constraint

$$D = \{x \in \mathbb{R}^n \mid a_i \leqslant x_i \leqslant b_i, i = 1, \dots, n\}$$

then the solution x^* of the following problem

$$\langle c, x\rangle \longrightarrow \max, \quad x \in D$$

has a form

$$x_i^* = \begin{cases} b_i, & \text{if } c_i > 0, \\ a_i, & \text{if } c_i \leqslant 0. \end{cases}$$

If D is $D = \{x \in \mathbb{R}^n \mid \|x - x^0\| \leqslant r\}$ then the problem

$$\langle c, x\rangle \longrightarrow \max, \quad x \in D$$

has the analytical solution x^* as

$$x^* = x^0 + \frac{c}{\|c\|} r.$$

Consider the convex maximization problem

$$f(x) \longrightarrow \max, \quad x \in D, \tag{2.1}$$

where $f:\mathbb{R}^n \to \mathbb{R}$ is a strongly convex and twice differentiable function and $D \subset \mathbb{R}^n$ is a simple set. Then the global optimality condition for this problem can be stated from the following theorem.

THEOREM 1 ([14]). *A point* $z \in D$ *with* $f'(z) \neq 0$ *is a solution of problem* (2.1) *if and only if the following condition holds*:

$$\langle f'(y), x - y \rangle \leqslant 0 \quad for\, all\, y \in E_{f(z)}(f) \quad and \quad x \in D,$$

where $E_c(f) = \{y \in \mathbb{R}^n \mid f(y) = c\}$. □

Define the auxiliary maximum function $\pi(y)$ as follows:

$$\pi(y) = \max_{x \in D} \langle f'(y), x - y \rangle, \qquad y \in \mathbb{R}^n \tag{2.2}$$

and the function $\theta(z)$ for $z \in D$:

$$\theta(z) = \max_{y \in E_{f(z)}(f)} \pi(y).$$

Since f is strongly convex, the set $E_{f(z)}(f)$ is compact. The auxiliary maximum function $\pi(y)$ is also known as the gap function in the literature [1, 7].

THEOREM 2. *Let* $z \in D$ *and* $f'(z) \neq 0$. *If* $\theta(z) = 0$ *then the point* z *is a solution to the problem* (2.1).

The proof is immediate from Theorem 1. □

A theoretical algorithm for the problem (2.1) based on the following problem

$$\pi(y) \longrightarrow \max, \qquad y \in E_{f(x^k)}(f) \tag{2.3}$$

was proposed in [3]. To make it numerically implementable the algorithm was adapted as follows in [4].

ALGORITHM ϵ-MAX
Input: A strongly convex function f and a simple set D, $\varepsilon_k > 0$ for all k and $\sum_{k=0}^{\infty} \varepsilon_k < +\infty$, and $\varepsilon > 0$.

Output: A global maximizer x of f over D.

Step 1. Choose a feasible solution $x^o \in D$ and a sequence $\{\varepsilon_k\}$ such that $f'(x^o) \neq 0$. Set $k := 0$.

Step 2. Find an ε_k solution $\bar{y}^k$ of the problem

$$\pi(y) \longrightarrow \max, \quad y \in E_{f(x^k)}(f)$$

that is, $\pi(\bar{y}^k) = \max_{x\in D}\langle f'(\bar{y}^k), x - \bar{y}^k\rangle \geqslant \max_{y\in E_{f(x^k)}(f)} \pi(y) - \varepsilon_k$ and $f(\bar{y}^k) = f(x^k)$. Let $\bar{x}^{k+1} \in D$ be a solution satisfying $\pi(\bar{y}^k) = \langle f'(\bar{y}^k), \bar{x}^{k+1} - \bar{y}^k\rangle$.

Step 3. If $\pi(\bar{y}^k) \leqslant \varepsilon - \varepsilon_k$ then output $x = x^k$ and terminate. Otherwise set $k := k+1$ and $x^k := \bar{x}^k$ return Step 2. □

The above algorithm used the set covering method [10] in Step 2 to find ϵ_k solution which is still hard and computationally available for only small dimensions.

THEOREM 3 ([4]). *Let* $f_* = \min_{x\in\mathbb{R}^n} f(x)$ *and* x^0 *be a feasible solution of problem* (2.1) *such that* $f(x^0) > f_*$. *Assume that* $f:\mathbb{R}^n \to \mathbb{R}$ *be strongly convex and continuously differentiable. Let a sequence* $\{\varepsilon_k\} \subset R$ *be such that* $\varepsilon_k > 0$ *for* $k = 0, 1, \ldots,$ *and*

$$\sum_{k=0}^{\infty} \varepsilon_k + f_* < f(x^0).$$

Then the sequence $\{x^k\}$ *generated by Algorithm* ϵ*-MAX is a maximizing sequence for problem* (2.1), *that is,*

$$\lim_{k\to\infty} f(x^k) = \max_{x\in D} f(x)$$

and every accumulation point of the sequence $\{x^k\}$ *is a global maximizer of* (2.1).

3. A Numerical Method for Solving the Convex Maximization

In order to develop an another method for solving problem (2.1), we will use some additional properties of the maximum function $\pi(y)$.

LEMMA 1 ([3]). *The function* $\pi(y)$ *is continuous on* $\mathbb{R}^n$.

THEOREM 4 ([3]). *The directional derivative of* $\pi(y)$ *at any point* $y \in \mathbb{R}^n$ *in any direction* $h \in \mathbb{R}^n$ *of the euclidian norm* 1 *exists, and is given by*

$$\pi'(y; h) = \max_{x\in D(y)} \langle f''(y)h, x\rangle - \langle f''(y)y + f'(y), h\rangle.$$

LEMMA 2. *If the set D is strictly convex, then the function $\pi(y)$ becomes differentiable and the gradient is given by*

$$\pi'(y) = f''(y)(x(y) - y) - f'(y),$$

where

$$\langle f'(y), x(y)\rangle = \max_{x \in D} \langle f'(y), x\rangle.$$

Proof. We write $\pi(y)$ in the form:

$$\pi(y) = \varphi(y) - \phi(y),$$

where $\varphi(y) = \max_{x \in D} \langle f'(y), x\rangle$ and $\phi(y) = \langle f'(y), y\rangle$. Since f is twice continuously differentiable, we can easily show that the function $\phi(y)$ is continuously differentiable on $\mathbb{R}^n$ and

$$\phi'(y) = f''(y)y + f'(y). \tag{3.1}$$

Thus we need only to show that $\varphi(y)$ is differentiable at a point $y \in \mathbb{R}^n$. First, define the vector function $p: \mathbb{R}^n \to \mathbb{R}^n$ by

$$p(y) = arg \ \max_{x \in D} \langle f'(y), x\rangle.$$

Also denote by $D(y)$ the set:

$$D(y) = \{x \in D : \langle f'(y), x\rangle = \varphi(y)\}, \quad y \in \mathbb{R}^n.$$

We note that since D is strictly convex then $D(y)$ consists of the unique point $p(y)$. We show that the function $p(y)$ is continuous. Suppose on the contrary that $p(y)$ is not continuous. Then there exist a point $\bar{y}$ and a sequence $\{y^k\} \subset \mathbb{R}^n$ such that

$$\lim_{k \to \infty} y^k = \bar{y},$$
$$\lim_{k \to \infty} p(y^k) = z,$$

and $p(\bar{y}) \neq z$. We have

$$\varphi(\bar{y}) = \max_{x \in D} \langle f'(\bar{y}), x\rangle = \langle f'(\bar{y}), p(\bar{y})\rangle.$$

On the other hand, due to continuity of $\varphi(y)$, we get

$$\varphi(\bar{y}) = \lim_{k \to \infty} \varphi(y^k) = \lim_{k \to \infty} \langle f'(y^k), p(y^k)\rangle = \langle f'(\bar{y}), z\rangle.$$

It means that $z \in D(\bar{y})$ and $p(\bar{y}) \in D(\bar{y})$ which contradicts that the set $D(\bar{y})$ contains a single point. Consequently, the vector function $p(y)$ is continuous on $\mathbb{R}^n$. By Theorem 4, the directional derivative of $\varphi(y)$ at any point $y \in \mathbb{R}^n$ in any direction $h \in \mathbb{R}^n$ of norm 1 exists, and is given by

$$\varphi'(y; h) = \langle f''(y)h, p(y)\rangle, \tag{3.2}$$

where $p(y) = arg\max_{x\in D}\langle f'(y), x\rangle$. Based on this formula, we show that the partial derivatives of the function $\varphi(y)$ at a point y exist and continuous. In order to do that evaluate right-side and left-side partial derivatives of the function with respect to ith components at the point y in the following:

$$\frac{\partial\varphi(y)}{\partial y_i}\Big|_+ = \lim_{\varepsilon\to+0}\frac{\varphi(y+\varepsilon e^i)-\varphi(y)}{\varepsilon} = \frac{\partial\varphi(y)}{\partial e^i} = [f''(y)p(y)]_i,$$
$$i = 1, 2, \dots, n,$$
$$\frac{\partial\varphi(y)}{\partial y_i}\Big|_- = \lim_{\varepsilon\to+0}\frac{\varphi(y-\varepsilon e^i)-\varphi(y)}{-\varepsilon} = -\frac{\partial\varphi(y)}{\partial(-e^i)} = [f''(y)p(y)]_i,$$
$$i = 1, 2, \dots, n.$$

Thus we conclude that right-side and left-side partial derivatives of the function at the point y are equal. This shows that the partial derivatives of the function exist and have the form:

$$\frac{\partial\varphi(y)}{\partial y_i} = [f''(y)p(y)]_i, \quad i = 1, 2, \dots, n.$$

Since $p(y)$ is continuous then these partial derivatives are also continuous functions. Hence, we conclude that the function $\varphi(y)$ is differentiable on $\mathbb{R}^n$ with the gradient $\varphi'(y)$:

$$\varphi'(y) = f''(y)p(y). \tag{3.3}$$

Taking into account (3.1)–(3.3), we have

$$\begin{aligned}\pi'(y) &= \varphi'(y) - \phi'(y) = f''(y)p(y) - f''(y)y - f'(y)\\ &= f''(y)(p(y) - y) - f'(y),\end{aligned}$$

which completes the proof. □

LEMMA 3. *If there is a point $y \in \mathbb{R}^n$ such that $\pi(y) > 0$ and $y \in E_{f(z)}(f)$ for a feasible point $z \in D$, then*

$$f(x(y)) > f(z)$$

holds, where $x(y) \in D$ satisfies $\langle f'(y), x(y)\rangle = \max_{x\in D}\langle f'(y), x\rangle$.

Proof. By the definition of $\pi(y)$, we have

$$\pi(y) = \max_{x \in D} \langle f'(y), x - y \rangle = \langle f'(y), x(y) - y \rangle.$$

Since f is strongly convex we have

$$f(u) - f(v) \geqslant \langle f'(v), u - v \rangle$$

for all $u, v \in \mathbb{R}^n$ [16]. Therefore, the assumption in the lemma implies that

$$f(x(y)) - f(z) = f(x(y)) - f(y) \geqslant \langle f'(y), x(y) - y \rangle = \pi(y) > 0. \qquad \square$$

LEMMA 4. *If* $\theta(z) = \pi(y^*) = 0$ *then* $f(z) = f(x(y^*))$.

Proof. Since f is convex, we have $f(u) - f(v) \geqslant \langle f'(v), u - v \rangle$ for all $u, v \in \mathbb{R}^n$. Therefore, setting $u = x(y^*)$ and $v = z$ in the latter gives us

$$\begin{aligned} f(x(y^*)) - f(z) &= f(x(y^*)) - f(y^*) \\ &\geqslant \langle f'(y^*), x(y^*) - y^* \rangle = \pi(y^*) = 0. \end{aligned}$$

Hence, $f(x(y^*)) \geqslant f(z)$. On the other hand, by Theorem 2, the point z is a global maximizer. Thus, $f(x(y^*)) = f(z)$. The proof is complete. $\square$

THEOREM 5. *Let* $z \in D$ *and* $y \in E_{f(z)}(f)$. *Assume that the vectors* $\pi'(y)$ *and* $f'(y)$ *are linearly independent. Then there is a point* $\bar{y} \in E_{f(z)}(f)$ *in a neighborhood of* y *such that* $\pi(\bar{y}) > \pi(y)$.

Proof. Let $y \in E_{f(z)}(f)$ i.e. $f(y) = f(z)$. Find a small variation δy which satisfies $\pi(y + \delta y) > \pi(y)$ and $y + \delta y + o(\|\delta y\|) \in E_{f(z)}(f)$. It is equivalent to the following problem to find δy such that [6]:

$$\delta\pi(\delta y) = \langle \pi'(y), \delta y \rangle > 0, \tag{3.4}$$

$$\delta f(\delta y) = \langle f'(y), \delta y \rangle = 0. \tag{3.5}$$

Since a set of δy satisfying condition (3.5) is a linear space, it is sufficient to find δy with $\langle \pi'(y), \delta y \rangle \neq 0$. In fact, if $\langle f'(y), \delta y \rangle = 0$ and $\langle \pi'(y), \delta y \rangle = 0$, then it follows that $f'(y)$ and $\pi'(y)$ are linear dependent which contradict the assumption of the lemma. Consequently, systems (3.4) and (3.5) has a nonzero solution δy. For all $\rho \in \mathbb{R}_+$, the first-order Taylor series expansion yields

$$f(y + \rho\delta y) = f(y) + \rho \langle f'(y), \delta y \rangle + \rho^2 o(\|\delta y\|^2).$$

Assume that $\|\delta y\| = 1$ and ρ be a small parameter. Now we introduce a small correction $\widetilde{y}$ as follows:

$$\|\widetilde{y}\| = o(\rho^2), \tag{3.6}$$

$$f(y + \rho\delta y + \widetilde{y}) - f(z) = 0. \tag{3.7}$$

Then according to ([6], pp. 399), we have

$$\begin{aligned}\pi(y + \rho\delta y + \widetilde{y}) &= \pi(y) + \rho\langle \pi'(y), \delta y\rangle + o(\rho^2)\\ &> \pi(y) + \rho\langle \pi'(y), \delta y\rangle.\end{aligned} \tag{3.8}$$

If we set $\bar{y} = y + \rho\delta y + \widetilde{y}$, then we obtain $\pi(\bar{y}) > \pi(y)$, $\bar{y} \in E_{f(z)}(f)$ which proves the assertion. □

LEMMA 5. *Let the set D be strictly convex and $z \in D$. If z is a local maximizer of the problem* (2.1) *then*

$$\pi'(z) = -f'(z).$$

Proof. First we show that $\pi(z) = 0$. Since z is a local maximizer then

$$\langle f'(z), x - z\rangle \leqslant 0$$

holds for all $x \in D$. Consequently,

$$\pi(z) = \max_{x \in D}\langle f'(z), x - z\rangle = 0.$$

According to Lemma 2, $\pi'(z)$ is computed as

$$\pi'(z) = f'(z)[x(z) - z] - f'(z),$$

where

$$\langle f'(z), x(z)\rangle = \max_{x \in D}\langle f'(z), x\rangle.$$

Clearly, $x(z) = z$. Hence, we obtain $\pi'(z) = -f'(z)$. □

Finding a point $y \in E_{f(z)}(f)$ is justified by the following statement.

LEMMA 6. *Let $f: \mathbb{R}^n \to \mathbb{R}$ be a strongly convex and differentiable function and $z \in \mathbb{R}^n$. If a vector $p \in \mathbb{R}^n$ satisfies $\langle f'(z), p\rangle < 0$ then there exists a positive number $\beta > 0$ such that $z + \beta p \in E_{f(z)}(f)$.*

Proof. Introduce the set at the point $z \in \mathbb{R}^n$

$$L(z, f) = \{x \in \mathbb{R}^n \,|\, f(x) \leqslant f(z)\},$$

which is compact due to strongly convexity of f [16]. Consequently, there is a number $M > 0$ for which $\|x\| \leqslant M$ holds for all $x \in L(z, f)$. Denote by $\phi(\alpha)$ the following function:

$$\phi(\alpha) = f(z + \alpha) - f(z).$$

Obviously, $\phi: \mathbb{R}_+ \to \mathbb{R}$ is convex. Write down Taylor's formula for the function f at z:

$$f(z + \alpha p) - f(z) = \alpha \langle f'(z), p \rangle + o(\alpha \|p\|),$$

where $\frac{o(\alpha\|p\|)}{\alpha} \to 0$ as $\alpha \to 0$. Since $\langle f'(z), p \rangle < 0$, there exists a sufficiently small $\epsilon \geqslant 0$ such that

$$f(z + \alpha p) - f(z) < 0, \quad \alpha \in (0, \epsilon],$$

in particular, $\phi(\epsilon) < 0$. Hence, we have

$$z + \alpha p \in L(z, f) \quad \text{for all } \alpha (0, \epsilon].$$

We can easily show that for a fixed number $\gamma > \epsilon$ the inequality

$$f(z + \alpha p) - f(z) \geqslant 0 \quad \text{for all } \alpha \in [\gamma, +\infty[$$

holds. Otherwise, we have

$$f(z + \alpha p) - f(z) < 0 \quad \text{for all } \alpha \in [\gamma, +\infty[.$$

If we construct a sequence of $\{\alpha_k\}$ in the following way:

$$\alpha_0 = \gamma, \qquad \alpha_1 = \gamma + 1, \ldots, \alpha_k = \gamma + k, \quad k = 0, 1, \ldots$$

then the sequence $x^k = z + \alpha_k p \in L(z, f)$. Clearly, $\|x^k\| \to \infty$ as $k \to \infty$, which contradicts boundedness of $L(z, f)$. Thus

$$f(z + \gamma p) - f(z) \geqslant 0.$$

If $f(z + \gamma p) - f(z) = 0$ then take β as $\beta = \gamma$. If $f(z + \gamma p) - f(z) > 0$, then $\phi > 0$. Since ϕ is continuous, there exists a point $\beta \in (\epsilon, \gamma)$ such that $\phi(\beta) = 0$ or $f(z + \beta p) = f(z)$ which implies $z + \beta p \in L(z, f)$ proving the assertion. □

COROLLARY 1. *If* $f(x) = \langle Cx, x\rangle + \langle d, x\rangle + q$ *is a strongly convex quadratic function then* β *can be easily found analytically as follows*:

$$\beta = -\frac{\langle 2Cz + d, p\rangle}{\langle Cp, p\rangle}.$$

COROLLARY 2. *When* D *is strictly convex then there exists a positive* β *such that*

$$z - \beta\pi'(z) \in E_{f(z)}(f).$$

Lemmas 2–6 give us some heuristic approach to iterative solution of problem (2.1). In fact, we start with an arbitrary feasible point $z^0 \in D$ and then find a point $y^0 \in \mathbb{R}^n$ such that $y^0 \in E_{f(z)}(f)$ and $\pi(y^0) > 0$. The next approximation point will be $z^1 = x(y^0)$. We will continue this iteration process until we can not get $\pi(y^k) > 0$. Then the proposed algorithm based on the above assertions is the following.

ALGORITHM 1.

Step 1. Let $z^0 \in D$ and $k := 0$.
Step 2. Find a point x^k as a solution of the problem

$$x^k : \langle f'(z^k), x\rangle \to \max, \quad x \in D.$$

Step 3. Compute $\pi(z^k)$

$$\pi(z^k) = \langle f'(z^k), x^k - z^k\rangle.$$

Step 4. If $\pi(z^k) > 0$ then $z^{k+1} := x^k$, $k := k + 1$ and go to Step 2.
Step 5. Find $y^k \in E_{f(z^k)}(f)$ such that $\pi(y^k) > 0$, where

$$\pi(y^k) = \langle f'(y^k), x(y^k) - y^k\rangle.$$

Set $z^{k+1} := x(y^k)$, $k := k + 1$ and go to Step 2.
Step 6. Otherwise, z^k is a global solution to problem (2.1).

THEOREM 6. *Let* D *be a polyhedral set. Assume that in Algorithm* 1

$$\pi(z^k) > 0 \quad \textit{for all } k = 0, 1, \ldots$$

Then the sequence $\{z^k,\ k = 0, 1, \ldots\}$ *generated by the algorithm converges to a global solution of problem* (2.1) *in a finite number of steps.*

The proof is immediate from the optimality condition in Theorem 2, Lemmas 2–6 and taking into account that the problem has a polyhedral constraint set with a finite number of local maximizers. □

4. Numerical Experiments

The proposed algorithm has been implemented in programming language Borland C++ v.5.02 on personal computer with *Pentium*300 Mhz processor and 64 MB RAM. The results are given in Tables 1–4. The list of problems was as follows.

Table 1.

Problem	Number of variables	Number of constraints	Computation time in (sec)	Number of improvements*
1	2	2	0.0549	1
2	20	10	0.3846	2
3	20	10	1.3187	2
4	20	10	0.3297	3
5	20	10	1.2637	4
6	20	10	1.5934	5
7	5	6	0.0549	4
8	6	9	0.0549	2
9	60	90	6.59	6
10	200	300	235.49	15
11	2	5	0.0549	1

*Number of improvements means the number of vertices found during the computation at which the value of the function π is positive.

Table 2.

Problem	Initial local maximum value	Approximate global maximum value	Actual global maximum value
1	2	4	4
2	226.048	394.750795	394.7506
3	716.04773	749.776825	884.75058
4	5320.956055	8695.01193	8695.01193
5	682.58444	754.750335	754.75062
6	1557.40290	5082.15058	5082.199895
7	0	17	17
8	0.03567	0.07134	0.07134
9	2.8243E+11	5.64859E+11	5.64859E+11
10	7.0697E+44	1.41393E+45	1.41393E+45
11	1.8	3.25	3.4

Table 3.

Problem	The minimum value	Improvement* (%)	Total error**(%)
1	0	50	0
2	0	42.74	0
3	0	3.81	15.25
4	0	38.80	0
5	0	9.56	0
6	0	69.35	0
7	−101.9	14.30	0
8	0	50	0
9	0	50	0
10	0	50	0
11	0	42.65	4.41

(*,**) The improvement and total error in a percent has been computed using the following formula

$$\text{Improvement} = \frac{M_\varepsilon - M_0}{M - m} * 100\%,$$
$$\text{Total error} = \frac{M - M_\varepsilon}{M - m} * 100\%,$$

where M denotes the actual global maximum value, M_ε is an approximate global maximum value obtained from the computation, M_0 is the initial local maximum value from which the computation starts and m denotes the unconstrained minimum value of the objective function.

Table 4.

Dimension	The largest eigenvalue	Time (sec)
3	6.371799	0.91
5	17.177	0.92
10	67.84318	0.96
20	270.49297	1.29
30	608.2483	1.30
40	1081.111	1.83
50	1689.076	2.72
60	2432.145	4.71
70	3310.318	6.94
80	4323.594	9.84
90	5471.873	12.65
100	6755.35	15.01

Problem 1.

$$\begin{aligned} \max \quad & (x) = 2x_1^2 + 4x_2^2 - 5x_1x_2 \\ \text{s.t.} \quad & 0 \leqslant x_1 \leqslant 1 \\ & 0 \leqslant x_2 \leqslant 1. \end{aligned} \tag{4.1}$$

Problems from 2 to 6 had the form [5]:

$$\begin{aligned} \max \quad & f(x) = \tfrac{1}{2}\sum_{i=1}^{20} \lambda_i (x_i - \alpha_i)^2 \\ \text{s.t.} \quad & Ax \leqslant b \\ & x \geqslant 0 \end{aligned}$$

$$A^T = \begin{pmatrix} -3 & 7 & 0 & -5 & 1 & 1 & 0 & 2 & -1 & 1 \\ 7 & 0 & -5 & 1 & 1 & 0 & 2 & -1 & -1 & 1 \\ 0 & -5 & 1 & 1 & 0 & 2 & -1 & -1 & -9 & 1 \\ -5 & 1 & 1 & 0 & 2 & -1 & -1 & -9 & 3 & 1 \\ 1 & 1 & 0 & 2 & -1 & -1 & -9 & 3 & 5 & 1 \\ 1 & 0 & 2 & -1 & -1 & -9 & 3 & 5 & 0 & 1 \\ 0 & 2 & -1 & -1 & -9 & 3 & 5 & 0 & 0 & 1 \\ 2 & -1 & -1 & -9 & 3 & 5 & 0 & 0 & 1 & 1 \\ -1 & -1 & -9 & 3 & 5 & 0 & 0 & 1 & 7 & 1 \\ -1 & -9 & 3 & 5 & 0 & 0 & 1 & 7 & -7 & 1 \\ -9 & 3 & 5 & 0 & 0 & 1 & 7 & -7 & -4 & 1 \\ 3 & 5 & 0 & 0 & 1 & 7 & -7 & -4 & -6 & 1 \\ 5 & 0 & 0 & 1 & 7 & -7 & -4 & -6 & -3 & 1 \\ 0 & 0 & 1 & 7 & -7 & -4 & -6 & -3 & 7 & 1 \\ 0 & 1 & 7 & -7 & -4 & -6 & -3 & 7 & 0 & 1 \\ 1 & 7 & -7 & -4 & -6 & -3 & 7 & 0 & -5 & 1 \\ 7 & -7 & -4 & -6 & -3 & 7 & 0 & -5 & 1 & 1 \\ -7 & -4 & -6 & -3 & 7 & 0 & -5 & 1 & 1 & 1 \\ -4 & -6 & -3 & 7 & 0 & -5 & 1 & 1 & 0 & 1 \\ -6 & -3 & 7 & 0 & -5 & 1 & 1 & 0 & 2 & 1 \end{pmatrix}$$

$$b = (-5\ 2\ -1\ -3\ 5\ 4\ -1\ 0\ 9\ 40)^T.$$

Problem 2.

$$\begin{aligned} \max \quad & f(x) = \tfrac{1}{2}\sum_{i=1}^{20} (x_i - 2)^2 \\ \text{s.t.} \quad & Ax \leqslant b \\ & x \geqslant 0. \end{aligned} \tag{4.2}$$

Problem 3.

$$\begin{aligned} \max \quad & f(x) = \tfrac{1}{2}\sum_{i=1}^{20} (x_i + 5)^2 \\ \text{s.t.} \quad & Ax \leqslant b \\ & x \geqslant 0. \end{aligned} \tag{4.3}$$

Problem 4.

$$\begin{aligned} \max \quad & f(x) = \tfrac{1}{2}\sum_{i=1}^{20} 20x_i^2 \\ \text{s.t.} \quad & Ax \leqslant b \\ & x \geqslant 0. \end{aligned} \tag{4.4}$$

Problem 5.

$$\begin{aligned} \max \quad & f(x) = \tfrac{1}{2}\sum_{i=1}^{20} (x_i - 8)^2 \\ \text{s.t.} \quad & Ax \leqslant b \\ & x \geqslant 0. \end{aligned} \tag{4.5}$$

Problem 6.

$$\begin{aligned} \max \quad & f(x) = \tfrac{1}{2}\sum_{i=1}^{20} i\,(x_i - 2)^2 \\ \text{s.t.} \quad & Ax \leqslant b \\ & x \geqslant 0. \end{aligned} \tag{4.6}$$

In the above problem the algorithm has found a solution

$$x = \begin{pmatrix} 0 & 0 & 0 & 0.41269 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.50545 & 0 & 0 & 7.30558 & 0 & 12.7467 & 0 & 18.02915 \end{pmatrix}^T$$

with the objective value 5082.199895 which improved a result in [5].

Problem 7.

$$\begin{aligned} \max \quad & f(x) = \tfrac{1}{2}x^T Qx - c^T x \\ \text{s.t.} \quad & 20x_1 + 12x_2 + 11x_3 + 7x_4 + 4x_5 \leqslant 40 \\ & 0 \leqslant x \leqslant 1, \end{aligned} \tag{4.7}$$

where $Q = 100 * I$; $c^T = (42, 44, 45, 47, 47, 5)$ and I *is a unit matrix.*

Problems 8–10 have been constructed using a technique proposed by Calamai et al. [2]. We had constructed the problems of the following type:

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} 3^{(j-L)}\left((x_j-1)^2+(y_j-1)^2\right) \\ \text{s.t.} \quad & A^1x+A^2y \leqslant b \\ & x \geqslant 0, \quad y \geqslant 0, \end{aligned}$$

where $A^1, A^2 \in \mathbb{R}^{n\times 3n}$, $x, y \in \mathbb{R}^n$ and $b \in \mathbb{R}^{3n}$ and if denote by a^1_{ij} and a^2_{ij} elements of A^1 and A^2 then they are defined as

$$\begin{aligned} & a^1_{j(3j-2)}=\alpha, \quad a^2_{j(3j-2)}=\beta, \quad b_{3j-2}=\alpha+\beta+\alpha\beta, \\ & a^1_{j(3j-1)}=1, \quad a^2_{j(3j-1)}=-(\beta+1), \quad b_{3j-1}=0, \\ & a^1_{j(3j)}=-(\alpha+1), \quad a^2_{j(3j)}=1, \quad b_{3j-1}=0 \end{aligned}$$

for $j=1,\dots,n$, where L is a positive integer parameter and $\alpha=\sqrt{3}$; $\beta=2$. The objective function attains the global maximum value $4\cdot\sum_{j=1}^{n}3^{(j-L)}$ at $x=(\underbrace{3,\dots,3}_{n},\underbrace{1,\dots,1}_{n})^T$. These problems have been tested for different dimensions.

Problem 8. $L=7,\ n=3$.

Problem 9. $L=7,\ n=60$.

Problem 10. $L=7,\ n=200$.

Problem 11 ([8]).

$$\begin{aligned} \max \quad & f(x)=(x_1-1.2)^2+(x_2-0.6)^2 \\ \text{s.t.} \quad & -2x_1+x_2 \leqslant 1 \\ & x_2 \leqslant 2 \\ & x_1+x_2 \leqslant 4 \\ & x_1 \leqslant 3 \\ & 0.5x_1-x_2 \leqslant 1 \\ & x_1 \geqslant 0, \quad x_2 \geqslant 0 \end{aligned} \tag{4.8}$$

Also, the problem of finding the largest eigenvalue of a positive definite symmetric matrix A was formulated as the following equivalent convex maximization problem ([13], Theorem 32.3, pp. 344):

$$f(x) = \langle Ax, x\rangle \to \max, \quad \|x\| \leqslant 1,$$

and then it has been numerically solved by the proposed algorithm. For this problem, we have

$$f'(x) = 2Ax, \quad f''(x) = 2A,$$

and the problem

$$\langle f'(x), y\rangle \to \ \max, \ \|x\| \leqslant 1$$

has the unique solution

$$x(y) = \frac{Ay}{\|Ay\|}.$$

Consequently,

$$\begin{aligned}\pi(y) &= \max_{\|x\| \leqslant 1} \langle 2Ay, x - y\rangle \\ &= 2\langle Ay, x(y)\rangle - 2\langle Ay, y\rangle \\ &= 2(\|Ay\| - \langle Ay, y\rangle),\end{aligned} \tag{4.9}$$

$$\pi' = f''(y)[x(y) - y] - f'(y) = \frac{2A^2 y}{\|Ay\|} - 4Ay.$$

In Table 4, we found the largest eigenvalue of the matrix A for different dimensions.

$$A = \begin{pmatrix} n & n-1 & n-2 & \dots & 1 \\ n-1 & n & n-1 & \dots & 2 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 2 & 3 & \dots & n \end{pmatrix}.$$

5. Conclusions

We have proposed a method for solving the convex maximization problem based on the global optimality conditions by Strekalovsky. Under some assumptions the proposed method had a convergence property. The computational results are given. Also, the problem of finding the largest eigenvalue of a matrix has been solved by the proposed algorithm.

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